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HW 4

MATH 534A
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1. Let $\langle x, y \rangle = \sum_{j=1}^n x^i y^j$ denote the usual inner product on $\mathbb{R}^n$ and suppose that A is a real symmetric $n \times n$ matrix. Let $Q(x) = \langle x, Ax \rangle$ denote the quadratic form associated with the matrix A defined for $x \in \mathbb{R}^n$.

   (a) Let $q(x)$ denote the restriction of $Q(x)$ to $x \in S^{n-1}$ the unit sphere. Show that the critical points for $q(x)$ on S^{n-1} are the eigenvectors for A (use Lagrange multipliers).

   Solution. Since A is real symmetric, it can be factored

$$A = R \Lambda R^T$$

where the columns of R, a unitary matrix, are the eigenvectors of A, Λ a diagonal matrix whose entries are the eigenvalues of A. Now, the quadratic form Q is

$$\begin{aligned} Q(x) &= x^T R \Lambda R^T x \\ &= x^T (R \Lambda R^T)^T x \\ &= x^T R^T \Lambda^T R x \\ &= (Rx)^T \Lambda (Rx) \\ &= y^T \Lambda y \\ &= \sum_{i=1}^n \lambda_i y_i^2 \end{aligned}$$

Now, $dq_p(v) = dQ_p(v)$ for $v \in T_p S^{n-1}$. To find the critical points for q, consider

$$dQ_p(v) = \lambda \langle p, v \rangle$$

for $p \in S^{n-1}$ and $v \in T_p \mathbb{R}^n$.

$$dQ_p(v) = 2 \sum_{i=1}^n \lambda_i y_i v_i = \lambda \sum_{i=1}^n p_i v_i.$$

Whatever λ is, the p_i are determined by the y_i, which are the components of the eigenvectors of A, by the definition $y = Rx$, the projection of x onto the eigenspace of A.

Therefore, the critical points of q are the eigenvectors of A.

(b) The gradient vector field ∇q associated with the function q on S^{n-1} is the vector field on S^{n-1} uniquely determined by the relation:

$$dq_p(v) = \langle \nabla q_p, v_p \rangle_p \text{ for all } v \in T_p S^{n-1}.$$

The inner product $\langle \cdot, \cdot \rangle_p$ is just the inner product that T_pS^{n-1} has as a consequence of being a subspace of $T_p\mathbb{R}^n \simeq \mathbb{R}^n$ with the inner product given above. Show that for $p \in S^{n-1}$,

$$\nabla q_p = \nabla Q_p - \langle \nabla Q_p, p \rangle p,$$

where ∇Q_p is defined by,

$$dQ_p(v) = \langle Q_p, v \rangle \text{ where } v \in T_pR^n$$

Note that if $p \in S^{n-1}$ is a critical point for q then the vector field ∇q is 0 at p and the constant function $x(t) = p$ is an integral curve for ∇q.

Solution. By definition,

$$dq_p(v) = \langle \nabla q_p, v \rangle.$$

The goal is to extract everything from ∇Q_p that is orthogonal to T_pS^{n-1}. For $v \in T_pS^{n-1}$,

$$dq_p(v) = dQ_p(v) = \langle \nabla Q_p, v \rangle.$$

Observe that, for points on S^{n-1}, $\langle p, v \rangle = 0$. Therefore, for $p \in S^{n-1}$, we have that

$$\nabla Q_p - \langle p, \nabla Q_p \rangle p$$

and subsequently

$$dq_p(v) = \langle \nabla Q_p - \langle p, \nabla Q_p \rangle p, v \rangle$$

and

$$\nabla q_p = \nabla Q_p - \langle p, \nabla Q_p \rangle p.$$

(c) Suppose that $t \to x(t) \in S^{n-1}$ is an integral curve for the vector field $-\nabla q$ on S^{n-1}. Show that the function $t \to q(x(t))$ is a strictly decreasing function of t unless $x(t)$ is an integral curve that just sits at one of the critical points of q. Hint: look at the derivative of $t \to q(x(t))$. What are the values of the function $q(x)$ at the critical points $x \in S^{n-1}$ for q? It is not hard to see that as $t \to \pm\infty$ the integral curve $x(t)$ must tend to a zero of the vector field $-\nabla q$. What then are the possible limiting values for $q(x(t))$ as $t \pm \infty$?

Solution.

$$\begin{aligned}\frac{d}{dt}q(x(t)) = d_{q(x(t))}(x'(t)) &= \langle \nabla q_{x(t)}, -\nabla q_{x(t)} \rangle \\ &= -\langle \nabla q_{x(t)}, \nabla q_{x(t)} \rangle\end{aligned}$$

But, $\langle \nabla q_{x(t)}, \nabla q_{x(t)} \rangle \geq 0$. Therefore,

$$\frac{d}{dt}q(x(t)) \leq 0$$

At the critical points, $\nabla q_{x(t)} = 0$, and therefore the function is unchanging. At all other points, the function $q(x(t))$ is decreasing.

(d) Define a vector field $X(p)$ on $\mathbb{R}^n \setminus \{0\}$ by,

$$X(p) = \nabla Q_p - \frac{\langle \nabla Q_p, p \rangle p}{\langle p, p \rangle} \text{ for } p \in \mathbb{R}^n \setminus \{0\}.$$

Show that if $t \to x(t)$ is an integral curve for the vector field X in $\mathbb{R}^n \setminus \{0\}$, that is,

$$\frac{dx}{dt} = X(x(t)),$$

and $x(0) \in S^{n-1}$, then for all t, $x(t) \in S^{n-1}$. Conclude that $x(t)$ is actually an integral curve for ∇q on S^{n-1}. Hint: calculate the derivative of $\langle x(t), x(t) \rangle$.

Solution. We need to show that the norm of x is unchanging in time. Therefore, if it starts on the unit sphere, it stays on the unit sphere for all time. Consider

$$\begin{aligned} \frac{d}{dt}\langle x(t), x(t) \rangle &= 2\langle x'(t), x(t) \rangle \\ &= \langle \nabla Q_{x(t)} - \frac{\langle \nabla Q_{x(t)}, x(t) \rangle x(t)}{\langle x(t), x(t) \rangle}, x(t) \rangle \\ &= 0 \end{aligned}$$

since the first entry in the inner product represents the portion of $\nabla Q_{x(t)}$ that is orthogonal to $x(t)$.

2. Define $e(\theta) = (\cos\theta, \sin\theta)$. Then $e : \mathbb{R} \to S^1$. Fix θ_0 and define a curve in S^1 through the point $p = e(\theta_0)$ by,

$$t \to e(\theta_0 + t).$$

The tangent vector at $p \in S^1$ associated with this curve at $t = 0$ is called $\left(\frac{\partial}{\partial \theta}\right)_p$. Show that this vector does not depend on the choice of θ_0 (more than one θ_0 can represent the same point p). Define a map $\pi : \mathbb{R}^2 \to S^1 \times S^1$ by:

$$\pi(\theta, \phi) = (e(\theta), e(\phi)).$$

Let α, β be real numbers and define a vector field V on $\mathbb{R}^2$ by,

$$V(p) = \alpha\left(\frac{\partial}{\partial x}\right)_p + \beta\left(\frac{\partial}{\partial y}\right)_p \tag{1}$$

(a) Show that if F_t is the one parameter flow associated with the vector field V then for any $p \in \mathbb{R}^2$, $\pi F_t(p)$ is an integral curve for the vector field,

$$\alpha\frac{\partial}{\partial \theta} + \beta\frac{\partial}{\partial \phi}, \tag{2}$$

defined on $S^1 \times S^1$.

Solution. We need to evaluate $\frac{d}{dt}\pi F_t(p)$ at $t=0$:

$$\begin{aligned}
\frac{d}{dt}\pi F_t(p)|_{t=0} &= \big(\frac{d\pi}{dt}\frac{dF_t(p)}{dt}\big)|_{t=0} \text{ by Chain Rule} \\
&= \big(\frac{d\pi}{dt}|_{t=0}\big)\big(\frac{dF_t(p)}{dt}|_{t=0}\big) \\
&= \big(\frac{d\pi}{dt}|_{t=0}\big)\big(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\big)
\end{aligned}$$

Differentiating π and changing variables on differentiation yields

$$\frac{d}{dt}\pi F_t(p)|_{t=0} \;=\; \alpha\frac{\partial}{\partial\theta}+\beta\frac{\partial}{\partial\phi}$$

and therefore $\pi F_t(p)$ is an integral curve for the vector field.

(b) What conditions on α,β guarantee that all the integral curves of (2) are periodic functions of t?

Solution. $\alpha,\beta\in\mathbb{Q}$ guarantees that all integral curves of (2) are periodic in t. This follows from the definition of periodicity

$$\theta(t+P)=\theta(t)$$

(c) Suppose that $(\alpha,\beta)=(M,N)$ where both M and N are integers. Fix the integral curve, $\gamma(t)=(e(Mt),e(Nt))$ for (2). Find all the integral curves of (1) which project under π onto γ. Find the closest distance between any two distinct integral curves for (1) which project onto γ.

Hint: the integral curves in question are lines $t\to tv+b_j$ for different values of b_j. The distance between two such lines (with b_1 and b_2) is $|\langle b_1-b_2,e\rangle|$ where e is a unit vector perpendicular to v. A little "number theory" is needed to finish up - look up the Euclidean algorithm and the greatest common divisor if you get stuck (you might also look at a small example).

Solution.

Remark: The reason to answer (c) is interesting is that the map π induces a diffeomorphism from $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$ to $S^1\times S^1$. A picture of the integral curve γ in $S^1\times S^1$ is obtained by looking at the intersection of all the lines in $\pi^{-1}\gamma$ with a "fundamental domain" $[0,2\pi)\times[0,2\pi)$. The closer the images get to one another the more "densely" does the integral curve wind around the torus.